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A THEORY OF ELASTIC STABILITY FOR INCOMPRESSIBLE, HYPERELASTIC BODIES

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Abstract—A linearized stability criterion for incompressible, hyperelastic bodies is derived from an exact criterion and applied to special problems. For an arbitrary body under dead loading corresponding to a uniform hydrostatic stress, sufficient conditions for stability are obtained for an arbitrary form of the strain energy function. In the case when the material is neo-Hookean, a general sufficient condition for stability is obtained for any traction boundary-value problem in which the stress is uniform throughout the body; and this result is used to get an estimate of the critical load for an Euler strut and for a column under tension.

INTRODUCTION

A NEW and general theory of stability for hyperelastic bodies under dead loading has been recently formulated by Beatty [1]. Beginning with a general energy criterion, he derives an exact dead load stability condition, called the D_{EM} criterion, from which he obtains a linearized stability condition, called the D_{LM} criterion. In either case, the equilibrium configuration whose stability is in question may be a finitely deformed configuration of the body. On the other hand, whereas the D_{EM} criterion is essentially applicable to both compressible and incompressible* materials for which there exists a strain energy function, the D_{LM} condition† is valid only for compressible materials. In Section 2 of the present paper, the linearized criterion for incompressible, hyperelastic bodies under dead loading is derived. Stability criteria for incompressible, elastic solids have also been formulated by Rivlin [3], Hill [4] and Green and Adkins [5], but the theory presented herein differs from these.

In all previous theories the superimposed displacement fields are required to satisfy the displacement boundary data and possibly a material constraint relation, such as the incompressibility condition; otherwise, the superimposed displacements are considered to be entirely arbitrary. In the present theory, however, an additional constraint is imposed. Specifically, from the class of all continuously differentiable virtual displacement fields which satisfy the boundary conditions of place, we admit as competing deformations for the criterion of stability only those which satisfy the incompressibility condition *and* the zero moment condition. The need for imposing the zero moment condition is made clear by the following example [1, 6].

* Of course, the class of virtual displacements must be narrowed to those which satisfy the incompressibility condition, but this constraint does not alter the form of the D_{EM} criterion.

† Truesdell and Noll [2] give a generalized form of the D_{LM} stability condition for which the existence of a stored energy function is not required. In the special case when there exists a strain energy function, their definition of stability reduces to the D_{LM} criterion derived in [1]. A similar generalization for incompressible materials may also be established.

Consider a column under compressive loading. Since a rigid body rotation about an axis which is not coincident with the axis of the column will violate the energy criterion, we should have to exclude the possibility of stability under compressive loading. This simple example shows that we cannot admit arbitrary geometrically possible, isochoric deformation fields; rather, in order to exclude undesirable instabilities of this type, a restriction on the rotational parts of the deformations is essential. Our example reminds us of the experimental difficulty of obtaining the Euler critical load; specifically, we recall that the experimenter very carefully arranges his apparatus so as to avoid destabilizing moments. Thus, motivated by physical experience, the condition we impose is that in the virtual configuration the resultant moment of the external forces is zero. In addition, we exclude as competing vector fields, uniform rigid body translations. Now Beatty [1] has shown that the zero moment condition excludes all infinitesimal rigid rotations, other than those which always yield a neutral stability of a trivial kind. For a column under compressive loading the zero moment condition admits rigid rotations about the axis of the strut, but no others; clearly, for such displacements the equilibrium configuration is neutrally stable.

In Section 4 the stability of an arbitrary incompressible, isotropic, hyperelastic body under dead loading corresponding to a uniform hydrostatic stress is examined. The case when the stress is a uniform hydrostatic tension has been investigated by Rivlin [3], Hill [4] and Green and Adkins [5]. Rivlin's results are restricted to the very special neo-Hookean material, whereas Hill's results are valid for the slightly more general Mooney–Rivlin material. Green and Adkins, on the other hand, obtain the solution for an arbitrary strain energy function. Because the theory developed here differs from those considered heretofore, in Section 4.1 we re-examine the case when the stress is a uniform hydrostatic tension. In addition, we shall investigate the stability of an arbitrary incompressible, isotropic body under a uniform hydrostatic pressure. Unlike the aforementioned problem where the zero moment condition serves only to restrict the class of virtual displacements, in order to obtain a sufficient condition for stability in the pressure case, we make full use of the zero moment condition and an inequality due to Korn [7–9]. In neither case do we assume a particular geometry for the body, nor do we assume a particular form for the strain energy function.

Finally, for a certain class of traction boundary-value problems in which the stress is uniform throughout the body, in Section 5 we derive a general sufficient condition for stability for a neo-Hookean material; and we obtain an estimate of the critical load for an Euler strut and for a column under tension.

1. NOTATION

So as to make this paper self-contained we begin by setting down the definitions of terms and the equations to be used in the subsequent development. For the most part the notation is identical with that employed in [1]. We use the usual indicial notation of Cartesian tensor analysis. A comma denotes partial differentiation; Greek suffices indicate quantities referred to a reference configuration C_0, and Latin suffices refer to quantities associated with the current configuration C_1. Thus $x_{i,\alpha} \equiv \partial x_i/\partial X_\alpha$ and $X_{\alpha,i} \equiv \partial X_\alpha/\partial x_i$, for example.

Let X_α denote the Cartesian coordinates of a material point $\mathbf{X}$ in a reference configuration C_0 of a body B having a material volume V_0 bounded by a material surface S_0. Let the body be deformed by the application of loads to an equilibrium configuration C_1 having a spatial volume V bounded by a surface S, and let x_i denote the Cartesian coordinates of the spatial point $\mathbf{x}$ in C_1. We assume all coordinate systems employed to be one and the same inertial Cartesian frame.

The motion of the material body carries the material points through various configurations or spatial positions, and we express this by the mappings $x_i = x_i(X_\alpha)$, or $X_\alpha = X_\alpha(x_i)$, where $x_i = X_i$ in C_0. We require that the mappings be single-valued and possess continuous partial derivatives with respect to their arguments to whatever order desired. In the neighborhood of a material point, we assume the mappings to be uniquely invertible; that is, $J = \det(x_{i,\alpha}) \neq 0$ nor ∞.

Let $\mathbf{f}(X_\alpha)$ denote the body force per unit mass, and let $\mathbf{T}(X_\alpha)$ be the surface force in C_1 per unit area in C_0. Since C_1 is an equilibrium configuration we have

$$\Sigma_{i\alpha,\alpha}+\rho_0 f_i = 0, \qquad \oint_{C_0} \mathbf{x}\times\mathbf{T}\,\mathrm{d}S_0+\int_{C_0}\mathbf{x}\times\mathbf{f}\,\mathrm{d}m = \mathbf{0}, \tag{1.1}$$

where $\Sigma_{i\alpha}$ is the asymmetric Kirchhoff stress tensor and ρ_0 is the density of mass m in C_0. If $\mathbf{\upsilon}$ is the unit outward directed normal vector to S_0, then the traction boundary conditions referred to C_0 are

$$T_i = \Sigma_{i\alpha}\upsilon_\alpha \tag{1.2}$$

on an assigned portion of S_0. Alternatively, equations (1.1) and (1.2) may be written

$$\sigma_{ij,j}+\rho f_i = 0, \tag{1.3}$$

and

$$t_i = \sigma_{ij}n_j \tag{1.4}$$

on an assigned portion of S corresponding to S_0. Here σ_{ij} is the symmetric Cauchy stress tensor, ρ is the density of mass in C_1, $\mathbf{n}$ is the unit outward directed normal vector to S, and $\mathbf{t}(X_\alpha)$ is the surface force in C_1 per unit area in C_1. In addition, we have the following relations [10, §26]:

$$\sigma_{ij} = J^{-1}x_{j,\alpha}\Sigma_{i\alpha}, \qquad J = \det(x_{i,\alpha}) = \rho_0/\rho, \tag{1.5}$$

where $x_{i,\alpha}$ denote the deformation gradients of the current configuration relative to the reference configuration.

Now consider a *virtual displacement* $\mathbf{v} = \mathbf{y}-\mathbf{x}$ from C_1 to a *virtual configuration* C_2 in which the spatial coordinates y_i are related to the material coordinates by a one-parameter family of motions $y_i = y_i(X_\alpha;\tau)$, $0 \leq \tau$, such that $y_i(X_\alpha;0) = x_i(X_\alpha)$; and assume that the loading $f_i^*(X_\alpha;\tau)$, the body force per unit mass, and $t_i^*(X_\alpha;\tau)$, the surface force per unit area in C_2, satisfy the dead loading conditions $f_i^*(X_\alpha;\tau) = f_i(X_\alpha)$ and $t_i^*(X_\alpha;\tau) = t_i(X_\alpha)$ for all $\tau \geq 0$. Moreover, let $\psi = \hat{\psi}(X_\alpha, x_{i,\alpha})$ denote any tensor quantity of arbitrary order. Then the change in ψ following the virtual displacement is given by

$$\Delta\psi = \hat{\psi}^*(X_\alpha, y_{i,\alpha})-\hat{\psi}(X_\alpha, x_{i,\alpha}). \tag{1.6}$$

In particular, for dead loading we have from (1.2) and (1.6)

$$\Delta T_i = \Delta\Sigma_{i\alpha}\upsilon_\alpha = 0 \tag{1.7}$$

on an assigned portion of S_0, and

$$\Delta \mathbf{f} = \mathbf{0} \tag{1.8}$$

in $V_0 + S_0$.

2. THE DEAD LOAD STABILITY CRITERIA

2.1 *Exact dead load stability criterion*

According to the theory developed in [1], an equilibrium configuration of a hyperelastic body is said to be D_{EM} stable for boundary conditions of place and dead load tractions if the inequality

$$\int_{C_0} (\Sigma_{i\alpha} v_{i,\alpha} - \Delta\Sigma)\, \mathrm{d}V_0 \leq 0 \tag{2.1}$$

holds for the class of virtual displacements which satisfy the boundary conditions of place and the *zero moment condition:*

$$\int_{C_0} (\mathbf{x}+\mathbf{v}) \times \Delta \mathbf{T}\, \mathrm{d}S_0 + \int_{C_0} \boldsymbol{\alpha}\, \mathrm{d}V_0 = \mathbf{0}. \tag{2.2}$$

In these relations the increments are defined by (1.6), $\Sigma = \hat{\Sigma}(X_\alpha, x_{i,\alpha})$ is the strain energy per unit initial volume V_0, $\alpha_i = \varepsilon_{ijk} v_{j,\alpha} \Sigma_{k\alpha}$ and ε_{ijk} is the usual permutation symbol. Alternatively, with (1.5) we have $\alpha_i = J\varepsilon_{ijk} v_{j,p} \sigma_{pk}$. Also, in the derivation of (2.1) and (2.2) use has been made of the dead loading conditions (1.8) and the equilibrium equations (1.1) [cf. 1].

In particular, for dead load tractions everywhere the first integral in (2.2) vanishes in accordance with (1.7), and the zero moment condition becomes

$$\int_{C_0} \varepsilon_{ijk} v_{i,\alpha} \Sigma_{j\alpha}\, \mathrm{d}V_0 = \int_{C_1} \varepsilon_{ijk} v_{i,p} \sigma_{jp}\, \mathrm{d}V = 0. \tag{2.3}$$

The D_{EM} criterion is not limited to small deformations in C_1, and it applies to all elastic materials for which there exists a strain energy as described above. However, if the material is incompressible, the aforementioned class of virtual deformations must be narrowed to admit only those which satisfy the incompressibility condition

$$\Delta J = 0. \tag{2.4}$$

Nevertheless, the D_{EM} criterion retains the form (2.1).

Finally, if we introduce the exact stress–strain relation for incompressible, hyperelastic bodies [2, 12],

$$\Sigma_{i\alpha} = \frac{\partial \Sigma}{\partial x_{i,\alpha}} - pJX_{\alpha,i}, \tag{2.5}$$

where p is an arbitrary hydrostatic pressure and $J(x_{i,\alpha}) = 1$ for all deformations in C_1, the dead load stability criterion (2.1) may be written in the form

$$\int_{C_0} \left(\frac{\partial \Sigma}{\partial x_{i,\alpha}} v_{i,\alpha} - p v_{i,i} - \Delta\Sigma \right) \mathrm{d}V_0 \leq 0. \tag{2.6}$$

2.2 *The linearized dead load stability criterion*

To obtain the linearized dead load stability criterion for incompressible materials, we expand $\Sigma^*(X_\alpha, y_{i,\alpha})$ in a power series in the virtual displacement gradients $v_{i,\alpha}$ about the equilibrium configuration C_1, neglect terms greater than second order in $v_{i,\alpha}$, and obtain

$$\Delta\Sigma - \frac{\partial \Sigma}{\partial x_{i,\alpha}} v_{i,\alpha} = \tfrac{1}{2} A_{i\alpha j\beta} v_{i,\alpha} v_{j,\beta}, \tag{2.7}$$

where

$$A_{i\alpha j\beta} = A_{j\beta i\alpha} = \frac{\partial^2 \Sigma}{\partial x_{i,\alpha} \partial x_{j,\beta}} \tag{2.8}$$

are the second-order elastic moduli of the material. In the same way we expand $J(y_{i,\alpha})$, use (2.4) and the well-known relation $\partial J/\partial x_{i,\alpha} = J X_{\alpha,i}$, and get

$$v_{i,i} = \tfrac{1}{2} v_{i,j} v_{j,i}. \tag{2.9}$$

Upon substituting (2.7) and (2.9) into (2.6), we reach

$$\int_{C_0} \tfrac{1}{2} A^*_{i\alpha j\beta} v_{i,\alpha} v_{j,\beta} \, \mathrm{d}V_0 \geq 0 \tag{2.10}$$

for stability of the equilibrium configuration C_1, wherein we have put

$$A^*_{i\alpha j\beta} = A^*_{j\beta i\alpha} = A_{i\alpha j\beta} + pJ X_{\alpha,j} X_{\beta,i}. \tag{2.11}$$

Moreover, it can be shown that with $\operatorname{div} \mathbf{v} = 0$, to first order in $v_{i,\alpha}$ the incremental stress is given by

$$\Delta\Sigma_{i\alpha} = A^*_{i\alpha j\beta} v_{j,\beta}; \tag{2.12}$$

consequently, (2.10) may be written

$$\int_{C_0} \tfrac{1}{2} \Delta\Sigma_{i\alpha} v_{i,\alpha} \, \mathrm{d}V_0 \geq 0. \tag{2.13}$$

The inequality (2.13) is equivalent to the criterion obtained by Hill [4, equation (28)]. Moreover, we note that (2.13) has the same form as the incremental stress criterion derived in [1] for compressible, hyperelastic bodies. It follows, therefore, that the general theorems on uniqueness and normal mode frequencies obtained in [1] apply also to incompressible materials; however, for displacement boundary-value problems the incremental stress (2.12) is determined only to within an arbitrary hydrostatic pressure.

For future use we derive an alternative form of (2.13) in terms of the symmetric Cauchy stress tensor $(1.5)_1$. With the aid of $(1.5)_1$ and the symmetric Green deformation tensor, defined by

$$C_{\alpha\beta} = x_{i,\alpha} x_{i,\beta}, \tag{2.14}$$

(2.5) and (2.8) may be written

$$\sigma_{ij} = 2J^{-1} x_{i,\mu} x_{j,\nu} \frac{\partial \Sigma}{\partial C_{\mu\nu}} - p\delta_{ij}, \tag{2.15}$$

$$A_{i\alpha j\beta} = 2\delta_{ij} \frac{\partial \Sigma}{\partial C_{\alpha\beta}} + 4 x_{i,\mu} x_{j,\nu} \frac{\partial^2 \Sigma}{\partial C_{\alpha\mu} \partial C_{\beta\nu}}. \tag{2.16}$$

Upon setting

$$c_{ijkl} = 4J^{-1}x_{i,\alpha}x_{j,\beta}x_{k,\mu}x_{l,\nu}\frac{\partial^2\Sigma}{\partial C_{\alpha\beta}\partial C_{\mu\nu}}, \tag{2.17}$$

and

$$B_{ikjl} = A^{*}_{i\alpha j\beta}x_{k,\alpha}x_{l,\beta} = \sigma_{kl}\delta_{ij}+p(\delta_{ij}\delta_{kl}+\delta_{kj}\delta_{li})+c_{ikjl}, \tag{2.18}$$

where

$$c_{ijkl} = c_{klij} = c_{ijlk} = c_{jikl}, \qquad B_{ijkl} = B_{klij}, \tag{2.19}$$

we find that (2.10) may be written

$$\int_{C_1} B_{ikjl}v_{i,k}v_{j,l}\,\mathrm{d}V \geq 0. \tag{2.20}$$

With

$$e_{ij} = \tfrac{1}{2}(v_{i,j}+v_{j,i}) \quad \text{and} \quad \omega_{ij} = \tfrac{1}{2}(v_{i,j}-v_{j,i}), \tag{2.21}$$

it is easy to see that (2.20) is equivalent to the form given by Hill [4, equation (30)]; namely,

$$\Phi \equiv \int_{C_1} \tfrac{1}{2}[\sigma_{kl}v_{i,k}v_{i,l}+2pe_{ij}e_{ij}+c_{ijkl}e_{ij}e_{kl}]\,\mathrm{d}V \geq 0. \tag{2.22}$$

2.3 *Definitions of types of stability*

Since we admit as competing deformations only those in a certain class, it proves convenient to introduce the following

Definition of the Class I: *The set of all continuously differentiable virtual displacement fields which satisfy the boundary conditions of place, the incompressibility condition (2.4) and the zero moment condition (2.2), or their appropriate equivalent forms, is called the Class* I.

As shown in [1], it is important to distinguish the various types of stability, and for this purpose various mneumonic devices are used. For the most part we shall employ the same notation* here; however, to avoid the possibility of confusing the criteria to be used here with those used in [1], we introduce the following definition.

Definition of D_{EI} and D_{LI} Stability: An equilibrium configuration is said to be D_{EI} [D_{LI}] stable if, for all $v \in I$, (2.1) [(2.13)] or its equivalent holds.

Excluding uniform rigid translations, we note that (2.1) [(2.13)] vanishes for $\mathbf{v} = \mathbf{0}$; if it vanishes for some $\mathbf{v} \neq \mathbf{0}$, but is non-positive [non-negative] for all $\mathbf{v} \in \mathrm{I}$, then the equilibrium is said to be *neutrally stable.* Sometimes it is convenient to distinguish neutral stability from ordinary stability, for which the vanishing of (2.1), or (2.13), implies $\mathbf{v} = \mathbf{0}$; but we shall make no use of ordinary stability here (cf. [1, 11]).

3. KORN'S INEQUALITY

The importance of Korn's inequality† in elastic stability analysis was first discovered by Holden [6], who used it to derive (general) sufficient conditions for D_{LM}FT stability in

* As in [1], suffices, such as FB for fixed boundaries, FT for fixed tractions and MB for mixed boundary conditions, are appended to the particular criterion designation to further assist in the classification of types of stability.

† A proof of Korn's inequality has been given by Friedrichs [7]. See also Eidus [13] and Mikhlin [14].

the case when the stress is either uniform or non-uniform throughout the body. From these conditions he shows that in the traction boundary-value problem the D_{LM} stability criterion predicts non-zero critical loads, and he obtains a safe estimate of the critical load for an Euler strut. His method reduces the stability computation to the simpler algebraic problem of determining when a quadratic form is positive definite.

Korn's inequality states that for an assigned vector field $\mathbf{v}$ which is continuously differentiable in a region V with boundary S and which satisfies certain side conditions, there exists a constant K, called Korn's constant, depending only on the region V, such that

$$\int_V \omega_{ij}^2 \, \mathrm{d}V \leq K \int_V e_{ij}^2 \, \mathrm{d}V, \tag{3.1}$$

where $\omega_{ij}^2 = \omega_{ij}\omega_{ij}$, $e_{ij}^2 = e_{ij}e_{ij}$, and ω_{ij}, e_{ij} are defined by (2.21).

The inequality (3.1) cannot hold for arbitrary vector fields; specifically, (3.1) is violated for rigid rotations. It can be shown [7] that rigid rotations are the only continuously differentiable vector fields for which $e_{ij} = 0$ while $\omega_{ij} \neq 0$. To exclude rigid rotations side conditions are imposed on the vector field $\mathbf{v}$ which divide the inequality into three cases. Here we shall be interested only in

Case 2:

$$\int_V \omega_{ij} \, \mathrm{d}V = 0, \qquad \mathbf{v} \in C^1. \tag{3.2}$$

The class of admissible regions, called Ω-domains, is carefully stated in [7]. It suffices here to note that this class includes bounded regions with corners and edges. Two recent papers by Bernstein and Toupin [8] and Payne and Weinberger [9] have been concerned with determining the value of Korn's constant for particular regions. For Case 2, the only three-dimensional region for which the value of Korn's constant is presently known is the sphere.

For the sphere Payne and Weinberger [9] have obtained the best possible constant $K = \frac{43}{13}$; their inequality (3.1) being sharp in the sense that they exhibit the field $\mathbf{v}$ for which equality holds. However, this $\mathbf{v}$ is not in Class I, and so for $\mathbf{v} \in \mathrm{I}$ the value of Korn's constant remains to be found. On the other hand, lower bounds for K are known [8].

For $\mathbf{v} = a(\mathbf{h} \cdot \mathbf{x})\mathbf{h} \times \mathbf{x}$, where $a =$ const. and $\mathbf{h} \cdot \mathbf{h} = 1$, we have $\operatorname{div} \mathbf{v} = 0$; and for this choice of $\mathbf{v}$ Bernstein and Toupin [8] find that (3.1) yields

$$K \geq 1 + \frac{4h_i h_j I_{ij}}{(\delta_{kl} - h_k h_l) I_{kl}}, \tag{3.3}$$

where $I_{ij} = \int_V x_i x_j \, \mathrm{d}V$. If the origin of coordinates is taken at the centroid of V, the foregoing vector field satisfies the condition (3.2); therefore, this example belongs to Case 2. Upon varying $\mathbf{h}$ they find that the right-hand side of (3.3) will have extrema when $\mathbf{h}$ is in the direction of the principal axes of the symmetric moment of inertia tensor I_{ij}, and thus the lower bound for K has a maximum value given by

$$K_l = 1 + \frac{4I_1}{I_2 + I_3}, \tag{3.4}$$

where $I_1 \geq I_2 \geq I_3$ are the principal values of I_{ij}.

4. STABILITY OF AN INCOMPRESSIBLE BODY UNDER HYDROSTATIC STRESS

The stability of an isotropic, incompressible, hyperelastic body under dead loading corresponding to a uniform hydrostatic tension $P > 0$ was first investigated by Rivlin [3] for the special neo-Hookean material, for which $\Sigma = \alpha(\mathrm{I}-3)$, where α is a material constant. The same problem was studied by Hill [4] for the slightly more general Mooney–Rivlin material, for which $\Sigma = \alpha(\mathrm{I}-3)+\beta(\mathrm{II}-3)$, where α, β are material constants; and he finds for the stability limit

$$P = 4(\alpha+\beta). \tag{4.1}$$

With $\beta = 0$ we arrive at Rivlin's result;*

$$P = 4\alpha = \frac{2E}{3}, \tag{4.2}$$

where E is Young's modulus. However, Rivlin has shown that for the same loading there exists several possible equilibrium states, some of which are stable and others which are not. For the all-round tension problem, the stability limit has also been obtained by Green and Adkins [5],† but for an arbitrary strain energy function. We shall review their solution subsequently.

4.1 *The stability analysis*

For isotropic, incompressible materials the strain energy is a function of the invariants of the Green deformation tensor (2.14): $\Sigma = \Sigma(X_\alpha, \mathrm{I}, \mathrm{II})$, where

$$\mathrm{I} = C_{\alpha\alpha}, \qquad \mathrm{II} = \tfrac{1}{2}(\mathrm{I}^2 - C_{\alpha\beta}C_{\beta\alpha}), \qquad \mathrm{III} = \det C_{\alpha\beta} = 1. \tag{4.3}$$

In this case, we have

$$\frac{\partial \Sigma}{\partial C_{\alpha\beta}} = \frac{\partial \Sigma}{\partial \mathrm{I}}\delta_{\alpha\beta} + \frac{\partial \Sigma}{\partial \mathrm{II}}(\mathrm{I}\delta_{\alpha\beta} - C_{\alpha\beta}), \tag{4.4}$$

and

$$\frac{\partial^2 \Sigma}{\partial C_{\alpha\beta}\partial C_{\mu\nu}} = A\delta_{\alpha\beta}\,\delta_{\mu\nu} + B(\delta_{\alpha\mu}\,\delta_{\beta\nu} + \delta_{\alpha\nu}\,\delta_{\beta\mu}) + C(C_{\mu\nu}\,\delta_{\alpha\beta} + C_{\alpha\beta}\,\delta_{\mu\nu}) + DC_{\mu\nu}C_{\alpha\beta}, \tag{4.5}$$

where

$$A = -2B + \frac{\partial^2 \Sigma}{\partial \mathrm{I}^2} + 2\mathrm{I}\frac{\partial^2 \Sigma}{\partial \mathrm{I}\,\partial \mathrm{II}} + \mathrm{I}^2\frac{\partial^2 \Sigma}{\partial \mathrm{II}^2}, \tag{4.6}$$

$$2B = -\frac{\partial \Sigma}{\partial \mathrm{II}}, \qquad C = -\left(\frac{\partial^2 \Sigma}{\partial \mathrm{I}\,\partial \mathrm{II}} + \mathrm{I}\frac{\partial^2 \Sigma}{\partial \mathrm{II}^2}\right), \qquad D = \frac{\partial^2 \Sigma}{\partial \mathrm{II}^2}.$$

* Hill [4, §6] has remarked that for the neo-Hookean material his result, $P = 4\alpha$, conflicts with Rivlin's conclusion that the undeformed state is stable unless $\frac{1}{3} < P/E < (\frac{1}{4})^{\frac{1}{3}}$. He thus implies that Rivlin's analysis is incorrect; but he does not clarify the difficulty. I have found that Rivlin's analysis is accurate; but one statement in [3] is particularly misleading. The difficulty to which I refer, and which I believe to be the origin of Hill's criticism, is the typographical transposition of the second and fourth inequalities in the last paragraph of the abstract in [3]. With this simple alteration it follows that the undeformed state is stable provided $P < 2E/3$, but other equilibrium states are possible. Thus the asserted difference between the results of Rivlin and Hill is artificial.

† Green and Adkins make no mention of the apparent disagreement among the results of Hill and Rivlin.

For a given finite deformation (2.14), equations (4.3)–(4.6) may be used to determine (2.15)–(2.18).

Let us consider now a body of arbitrary shape subjected to a uniform hydrostatic stress $\sigma_{ij} = P\,\delta_{ij}$, where for tension $P > 0$ and for pressure $P < 0$. We assume that the equilibrium configuration is a state of pure, homogeneous deformation given by $x_i = \lambda X_i$, $\lambda > 0$. We note that in absence of body forces the equilibrium equations (1.3) are satisfied.

Since $x_{i,\alpha} = \lambda\,\delta_{i\alpha}$, (2.14) yields $C_{\alpha\beta} = \lambda^2\,\delta_{\alpha\beta}$; and from (4.3) we get $\lambda = 1$, $\mathrm{I} = \mathrm{II} = 3$, which corresponds to the undeformed state of the body in the stressed configuration C_1. Use of these results in (4.4), (4.5) and (4.6) leads to

$$\frac{\partial \Sigma}{\partial C_{\alpha\beta}} = (\alpha+2\beta)\delta_{\alpha\beta}, \tag{4.7}$$

and

$$\frac{\partial^2 \Sigma}{\partial C_{\mu\nu}\,\partial C_{\alpha\beta}} = \gamma\delta_{\mu\nu}\,\delta_{\alpha\beta} - \tfrac{1}{2}\beta(\delta_{\mu\alpha}\,\delta_{\nu\beta} - \delta_{\alpha\nu}\,\delta_{\beta\mu}), \tag{4.8}$$

where

$$\alpha = \left.\frac{\partial \Sigma}{\partial \mathrm{I}}\right|_{\lambda=1}, \qquad \beta = \left.\frac{\partial \Sigma}{\partial \mathrm{II}}\right|_{\lambda=1}, \qquad \gamma = (A+2C+D)\Big|_{\lambda=1} \tag{4.9}$$

which, when substituted into (2.15)–(2.17), yield*

$$\sigma_{ij} = (2\alpha+4\beta-p)\,\delta_{ij} = P\,\delta_{ij}, \tag{4.10}$$

and

$$c_{ijkl} = 4\gamma\,\delta_{ij}\,\delta_{kl} - 2\beta(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}). \tag{4.11}$$

From (4.10) we get

$$p = 2\alpha + 4\beta - P. \tag{4.12}$$

Finally, upon substituting (4.8) and (4.11) into (2.22), we find

$$\Phi = \int_{C_1} [(4\alpha+4\beta-P)e_{ij}^2 + P\omega_{ij}^2]\,\mathrm{d}V, \tag{4.13}$$

while the zero moment condition (2.3) yields

$$2P\int_{C_1} \omega_{ij}\,\mathrm{d}V = 0. \tag{4.14}$$

Now for various values of the loading the equilibrium configuration may be either stable, unstable, or neutrally stable according as $\Phi > 0$ for all $\mathbf{v}$, $\Phi < 0$ for some $\mathbf{v}$, or $\Phi = 0$ for some $\mathbf{v} \neq \mathbf{0}$ but is non-negative for all $\mathbf{v}$, respectively, where $\mathbf{v} \in \mathrm{I}$. We seek sufficient conditions for which $\Phi \geq 0$ for all $\mathbf{v} \in \mathrm{I}$.

Consider the case when the body is under a uniform all-around tension $P > 0$. From (4.13) it is apparent that a sufficient condition for D_{LI}FT stability is

$$0 < P \leq 4(\alpha+\beta). \tag{4.15}$$

* Our result (4.11) for the elasticities c_{ijkl} does not agree with that given by Hill [4] for the Mooney–Rivlin material ($\gamma = 0$); his elasticities do not satisfy the symmetry conditions $(2.19)_1$ [4, p. 238]. To compare the results here with those in [4], replace α and β in [4] by $2\alpha+4\beta$ and -2β, respectively.

Thus, for an arbitrary strain energy function, the D_{LI} stability limit for an arbitrary body under hydrostatic tension $P > 0$ is

$$P = \frac{2E}{3}, \tag{4.16}$$

where $E = 6(\alpha+\beta)$ is Young's modulus.*

From (4.13) we observe that $E > 0$ is necessary and sufficient for D_{LI} stability of the stress-free state.

We notice that our conclusion (4.16) is exactly the *same* as the results (4.1) and (4.2) obtained by Rivlin [3] and Hill [4] for very special strain energy functions; and we emphasize that our solution (4.15) is the same as that first obtained by Green and Adkins [5] with the aid of a different theory and in different notation. The subsequent analysis shows that $P > 0$ is not necessary for D_{LI}FT stability.

Let us now turn to the more practical case when the body is subjected to a uniform hydrostatic pressure $P = -\pi, \pi > 0$. We see that the zero moment condition (4.14) corresponds to (3.2); therefore, in (4.13) we may use Korn's inequality (3.1). We find

$$\Phi = \left(\frac{2E}{3}+\pi\right) \int_{C_1} e_{ij}^2 \,\mathrm{d}V - \pi \int_{C_1} \omega_{ij}^2 \,\mathrm{d}V \geq \left[\frac{2E}{3} - \pi(K-1)\right] \int_{C_1} e_{ij}^2 \,\mathrm{d}V. \tag{4.17}$$

From (4.17) it follows at once that a sufficient condition† for D_{LI}FT stability is

$$\pi \leq \frac{2E}{3(K-1)}. \tag{4.18}$$

If we take as an estimate of K the lower bound (3.4), we have the following critical load

$$\pi_{\mathrm{cr}} = E(I_2+I_3)/6I_1, \tag{4.19}$$

where $I_1 \geq I_2 \geq I_3$. In particular, for a sphere we get $\pi_{\mathrm{cr}} = E/3$, and for a circular column of length L and radius r we find

$$\pi_{\mathrm{cr}} = \begin{cases} E/\mu & \text{for} \quad \mu \geq 3, \\ E(1+\mu/3)/6 & \text{for} \quad \mu \leq 3, \qquad \mu = L^2/r^2. \end{cases} \tag{4.20}$$

We must emphasize that all of the foregoing stability limits are obtained for dead loading and may differ considerably from those which would obtain from a criterion appropriate for pressure loading.

4.2 *Comparison with Holden's analysis*

It is interesting to compare our result (4.18) with the result that may be obtained on the basis of Holden's analysis [6]. In the case when the stress tensor is uniform throughout

* For a simple elongation of a rod [5, p. 299], we may define a Young's modulus by the relation $E = \left.\frac{\mathrm{d}\sigma_{11}}{\mathrm{d}\lambda}\right|_{\lambda=1}$, where λ is the stretch in the direction of σ_{11}. A simple computation shows that $E = 6(\alpha+\beta)$.

† Suppose we have D_{LI}FT stability. Then in (4.17) $\Phi \geq 0$ for all $\mathbf{v} \in \mathrm{I}$. Therefore, if there exists a $\mathbf{v}_0 \in \mathrm{I}$ such that the equality in (3.1) holds, then clearly (4.18) is also a necessary condition. Unfortunately, we cannot guarantee the existence of $\mathbf{v}_0$. Nevertheless, I suspect that (4.18) is also necessary for D_{LI}FT stability.

the body, Holden shows [6, equation (3.22)] that

$$\int_{C_1} \sigma_{ij} v_{k,i} v_{k,j} \, dV \geq (t_3 - t)(K+1) \int_{C_1} e_{ij}^2 \, dV \tag{4.21}$$

provided that $t_3 - t < 0$, where

$$0 \leq t = \max\left\{\frac{(t_1-t_2)^2}{t_1+t_2}, \frac{(t_2-t_3)^2}{t_2+t_3}, \frac{(t_3-t_1)^2}{t_3+t_1}, 0\right\} \tag{4.22}$$

when the principal stresses $t_1 \geq t_2 \geq t_3$ are such that $t_i \neq -t_j, i \neq j$. When $t_i = -t_j$, $i \neq j$, t is redefined to exclude the term $(t_i - t_j)^2/(t_i + t_j)$. Use of the inequality (4.21) in (2.22) gives a sufficient condition for stability of a body subject to a uniform stress field, namely

$$\Phi \geq \int_{C_1} [(t_3 - t)(K+1)e_{ij}^2 + 2pe_{ij}^2 + c_{ijkl} e_{ij} e_{kl}] \, dV \geq 0. \tag{4.23}$$

In particular, when the stress is a uniform hydrostatic pressure we have $t = 0$, $t_3 = -\pi, \pi > 0$. Now with the aid of (4.11) and (4.12) it can be shown that Holden's analysis leads to the same result that we have found, namely (4.18). Since for hydrostatic tension $t_3 - t > 0$, we cannot use (4.23).

5. STABILITY OF A NEO-HOOKEAN BODY

In this section we investigate the stability of a neo-Hookean material in the dead load traction boundary-value problem when the stress is uniform throughout the body, and we find the critical load for an Euler strut and for a column under tension.

5.1 *A sufficient condition for* D_{LI}FT *stability*

With the aid of (4.5), (4.6) and (2.17), it is easy to show that $c_{ijkl} = 0$. Thus, (4.23) reduces to

$$\Phi \geq \int_{C_1} [(t_3 - t)(K+1) + 2p] e_{ij}^2 \, dV. \tag{5.1}$$

Consequently, from (5.1) it follows immediately that in any traction boundary-value problem where the principal stresses are uniform throughout the body and satisfy the conditions stated in Section 4.2, a sufficient condition for D_{LI}FT stability of a neo-Hookean body is

$$(t_3 - t)(K+1) + 2p \geq 0. \tag{5.2}$$

Therefore, the critical load we have found is determined by

$$t - t_3 = \frac{2p}{K+1}. \tag{5.3}$$

5.2 *Stability of an Euler column*

Consider a column with length L and a uniform cross-sectional area A in the configuration C_1, subjected to a compressive load $P > 0$ directed along the axis of the column such that

$$t_1 = t_2 = 0 \quad \text{and} \quad t_3 = -P/A. \quad \text{Then } t = 0. \tag{5.4}$$

The constitutive equation for a neo-Hookean body, which may be found in [3, 12] or easily derived from (2.15), is given by

$$t_k = -p + 2\alpha\lambda_k^2, \tag{5.5}$$

where $t_k = \sigma_{kk}$ (no sum). For simple compression, defined by $x_i = \lambda_i X_i$ (no sum), where $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda$, the incompressibility condition $(4.3)_3$ gives $\lambda_1^2 = 1/\lambda$. Thus (5.4) and (5.5) yield

$$p = \frac{2\alpha}{\lambda} \quad \text{and} \quad t_3 = -\frac{P}{A} = 2\alpha\left(\lambda^2 - \frac{1}{\lambda}\right). \tag{5.6}$$

Putting (5.4) and $(5.6)_1$ into (5.3), we find the relation from which the critical load may be found; namely,

$$P_{cr} = \frac{4\alpha A}{\lambda_{cr}(K+1)} = 2\alpha A\left(\frac{1}{\lambda_{cr}} - \lambda_{cr}^2\right), \tag{5.7}$$

where λ_{cr} is the value of the compression at the critical load. From (5.7) we get

$$\lambda_{cr} = \left(\frac{K-1}{K+1}\right)^{\frac{1}{3}} \tag{5.8}$$

hence,

$$P_{cr} = \frac{2AE}{3(K-1)}\left(\frac{K-1}{K+1}\right)^{\frac{2}{3}}, \tag{5.9}$$

where $E = 6\alpha$.

In particular, for a circular cylinder of radius r in the deformed state, if we take (3.4) as an estimate of K, we find

$$K-1 = \begin{cases} 2\mu/3 & \text{for} \quad \mu \geq 3, \\ 4/(1+\mu/3) & \text{for} \quad \mu \leq 3, \qquad \mu = L^2/r^2. \end{cases} \tag{5.10}$$

Therefore, (5.8) and (5.9) become

$$\lambda_{cr} = \begin{cases} (1+3/\mu)^{-\frac{1}{3}} & \text{for} \quad \mu \geq 3, \\ 2^{\frac{1}{3}}(3+\mu/3)^{-\frac{1}{3}} & \text{for} \quad \mu \leq 3, \end{cases} \tag{5.11}$$

and

$$P_{cr} = \begin{cases} (AE/\mu)(1+3/\mu)^{-\frac{2}{3}} & \text{for} \quad \mu \geq 3, \\ (AE2^{\frac{2}{3}}/6)(1+\mu/3)(3+\mu/3)^{-\frac{2}{3}} & \text{for} \quad \mu \leq 3. \end{cases} \tag{5.12}$$

We note that in none of the foregoing relations is the geometry restricted to that of a long, slender column, and in each case the geometry corresponds to the deformed state. To relate r and L of the deformed state to the corresponding quantities r_0 and L_0 in the reference configuration we use the relations $L = \lambda L_0, r = r_0/\sqrt{\lambda}$. In the case when $\mu \geq 3$, for example, we find (in obvious notation)

$$\lambda_{cr} = (1 - 3/\mu_0)^{\frac{1}{3}}, \qquad P_{cr} = (A_0 E/\mu_0)(1-3/\mu_0)^{-\frac{2}{3}}, \qquad \mu_0 \geq 6. \tag{5.13}$$

Now, as is well-known, the results of the classical buckling theory indicate that for a sufficiently long and slender column, the first deformation will be infinitesimal, in which case the geometry of the deformed state may be approximated by that in the reference configuration. It seems reasonable to expect that the same is true for rubber-like materials; indeed, we observe that (5.8) is consistent with this expectation. If we take as an estimate of K the lower bound (3.4), it is clear that for any long, slender column $K \gg 1$ [cf. (5.10) for example]. Thus (5.8) gives a value of $\lambda_{cr} \cong 1$; and (5.9) yields the following estimate of the critical load:

$$P_{cr} = 2AE/3(K-1), \tag{5.14}$$

where the geometry is that of the reference configuration.

We observe that (5.14) is the same as (4.18). Thus we have the interesting result that when the material is neo-Hookean, the critical load of a long, slender column subjected to a uniform hydrostatic pressure is very nearly the same as the critical load of an identical column under uniaxial compression.

Finally, for the special case of a long, slender circular cylinder (5.12) gives a critical load

$$P_{cr} = \pi E r^4/L^2. \tag{5.15}$$

It is of interest to see how the value (5.15) compares with the classical Euler load $P^*_{cr} = \pi^3 E r^4/4L^2$. We find

$$\frac{P_{cr}}{P^*_{cr}} = \frac{4}{\pi^2} \cong 0{\cdot}4; \tag{5.16}$$

so our estimate is safe.

5.3 *Stability in simple tension*

Consider a column subjected to a tensile load $P > 0$ directed along the axis of the column such that

$$t_1 = P/A \quad \text{and} \quad t_2 = t_3 = 0. \quad \text{Then} \quad t = P/A. \tag{5.17}$$

In this case the constitutive equation (5.5) for a neo-Hookean body yields

$$p = \frac{E}{3\lambda} \quad \text{and} \quad t_1 = \frac{P}{A} = \frac{E}{3}\left(\lambda^2 - \frac{1}{\lambda}\right), \tag{5.18}$$

where $\lambda_1 = \lambda, \lambda_2 = \lambda_3 = 1/\sqrt{\lambda}$. Since $t_3 - t < 0$ we may apply (5.3); we find

$$\lambda_{cr} = \left(\frac{K+3}{K+1}\right)^{\frac{1}{3}}, \qquad P_{cr} = \frac{2AE}{3(K+3)}\left(\frac{K+3}{K+1}\right)^{\frac{2}{3}} \tag{5.19}$$

for the critical state of a column under tension.

In particular, for a circular column we get an estimate

$$\lambda_{cr} = \begin{cases} [(1+6/\mu)/(1+3/\mu)]^{\frac{1}{3}} & \text{for} \quad \mu \geq 3, \\ 2^{\frac{1}{3}}[(2+\mu/3)/(3+\mu/3)]^{\frac{1}{3}} & \text{for} \quad \mu \leq 3; \end{cases} \tag{5.20}$$

and

$$P_{cr} = \begin{cases} \dfrac{AE}{\mu(1+6/\mu)}\left(\dfrac{1+6/\mu}{1+3/\mu}\right)^{\frac{2}{3}} & \text{for} \quad \mu \geq 3, \\ \dfrac{AE2^{-\frac{1}{3}}(1+\mu/3)}{3(2+\mu/3)}\left(\dfrac{2+\mu/3}{3+\mu/3}\right)^{\frac{2}{3}} & \text{for} \quad \mu \leq 3. \end{cases} \tag{5.21}$$

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Résumé—Un critère de stabilité linéarisé, pour corps incompressibles hyper-élastiques, est tiré d'un Critère exact et appliqué aux problèmes spéciaux. Pour un corps arbitraire soumis à une charge morte correspondant à une tension uniforme hydrostatique, des conditions suffisantes de stabilité sont obtenues pour une forme arbitraire de la fonction d'énergie de travail. Au cas où la matière serait néo-Hookénne une condition de stabilité générale et suffisante est obtenue pour tout problème de traction à valeur limite dans lequel la tension est uniforme dans tout le corps; et ce résultat est employé pour obtenir une évaluation de la charge critique pour une poutre d'Euler et pour une colonne soumise à une tension.

Zusammenfassung—Ein linearisiertes Stabilitäts-Kriterium für unzusammendrückbare hyperelastische Körper wird von einem genauen Kriterium abgeleitet und für besondere Probleme angewandt. Für einen beliebigen Körper unter ruhender Last, die einer gleichmäßigen hydrostatischen Spannung entspricht werden genügend Stabilitätsbedingungen erhalten um eine beliebige Form von Spannungsenergie-Funktion zu erzielen. Wenn das Material neu-Hookeisch ist wird für die Stabilität eine genügende Bedingung erhalten für alle Randwertsprobleme wenn die Spannung des Körpers gleichmässig ist; dieses Resultat wird angewandt um die kritische Last für einen Euler'schen Druckstab und für eine Säule unter Spannung zu erhalten.

Абстракт—Критерий линеаризованной устойчивости для несжимаемых, гиперэластичных тел производится из точного критерия и применяется к специальным проблемам. Для произвольного тела под мёртвой нагрузкой, соответствующей однородному гидростатическому напряжению получены удовлетворительные условия устойчивости для произвольной формы функции энергии напряжения. В случае, когда материал neo-Hookean достаточное условие для устойчивости получено для всякой проблемы силы сцепления граничного значения, в которой напряжение (однородно) равномерно по всему телу; и этот результат применяется для получения расчёта для критической нагрузки Эйлеровой балки и для колонны под напряжением.